STRESSES IN A WEDGE AS A RESULT OF SYMMETRICALLY AND ANTISYMMETRICALLY DISTRIBUTED LOAD AND TEMPERATURE

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The problem of determining the thermoelastic stresses from a linear heat source located at the apex of a wedge reduces to the basic problem of the theory of elasticity with external forces presented in the form of several functional series. The stresses are determined by means of the complex Kolosov–Muskhelishvili representation.

Let us examine the quasistatic problem of the distribution of thermal stresses in an infinite wedge with adiabatic boundaries and a divergence angle $2\psi \neq \Pi$. For definiteness, we consider the case of a plane stressed state. We assume that the thermophysical coefficients of the material do not exceed the limits of elasticity and are independent of temperature. Let a linear heat source whose intensity varies as $q \exp(i\omega t)$ be applied at the apex of the wedge $(r \geq 0; -\psi \leq \varphi \leq \psi)$. We know [3] that the temperature in a steady-state periodic regime is equal to

$$T = \frac{q}{2\lambda\psi} \exp(i\omega t) K_0(pr). \tag{1}$$

Here $p = \sqrt{(\omega/a)i}$; ω and q are constants.

The steady-state periodic thermal stresses are determined [3, 4] as the sum of the stresses

$$\sigma_\varphi = \bar{\sigma}_\varphi + \bar{\bar{\sigma}}_\varphi,\ \sigma_r = \bar{\sigma}_r + \bar{\bar{\sigma}}_r,\ \tau_{r\varphi} = \bar{\tau}_{r\varphi} + \bar{\bar{\tau}}_{r\varphi}. \tag{2}$$

The stresses with a single overscore are determined through the thermoelastic potential of the displacements θ from the formulas

$$\bar{\sigma}_\varphi = -2G\frac{\partial^2\theta}{\partial r^2},\ \bar{\sigma}_\varphi + \bar{\sigma}_r = -2G\Delta\theta, \tag{3}$$

$$\tau_{r\varphi} = 2G\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\theta}{\partial\varphi}\right]. \tag{4}$$

The stresses with two overscores are determined here by means of the Airy function in complex form, and the boundary conditions for this function, given an absence of external loads, are found from the equation

$$\bar{\bar{\sigma}}_\varphi + i\bar{\bar{\tau}}_{r\varphi} = -(\bar{\sigma}_\varphi + i\bar{\tau}_{r\varphi}) \text{ when } \varphi = \pm\psi. \tag{5}$$

Proceeding from the equation

$$\theta_t' = (1+\mu)\alpha a T,$$

we determine the thermoelastic potential of the displacements

$$\theta = A_1 K_0(pr), \quad A_1 = \frac{(1+\mu)\alpha a q}{2i\lambda\psi\omega}\exp(i\omega t). \tag{6}$$

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The stresses with one overscore, according to (3) and (4), have the form

$$\bar{\sigma}_\varphi + \bar{\sigma}_r = -AK_0(pr), \quad \bar{\sigma}_\varphi = -Af(pr), \quad \bar{\tau}_{r\varphi} = 0, \tag{7}$$

where

$$f(\rho r) = K_0(pr) + \frac{K_1(pr)}{pr}; \quad A = 2Gp^2A_1.$$

Considering the familiar series for the Bessel functions, we obtain the boundary conditions for the stress function in the form

$$\bar{\bar{\sigma}}_\varphi + i\bar{\bar{\tau}}_{r\varphi} = A\sum_{n=0}^{\infty}\rho^{2n}\{\alpha_n \ln\rho + \beta_n\} + A(2\rho)^{-2} \text{ when } \varphi = \pm\psi. \tag{8}$$

Here we denote

$$\alpha_n = -\frac{2n+1}{2n!\,(n+1)}; \quad \beta_n = \frac{(2n+1)\,\chi(n+1) - \chi(n+2)}{4n!\,(n+1)!}; \quad \rho = \frac{pr}{2};$$

$$\chi(n+1) = -\gamma^* + 1 + \frac{1}{2} + \ldots + \frac{1}{n}; \quad \gamma^* = 0.577\ \ldots$$

which is the Euler constant. In the boundary conditions the term in the form of $(2\rho)^{-2}$ produces no thermoelastic stresses and, consequently, it can be eliminated. We have derived a representation of the boundary conditions in the form of several functional series which are generalizations of the power series.

This is the problem which we find in studying a linear heat source at the apex of a wedge when we consider the transfer of heat to the side surfaces [1] in a steady-state regime:

$$T = \frac{q}{2\psi\lambda l}K_0(mr), \quad \bar{\sigma}_\varphi = -Ef(mr),$$

$$\bar{\sigma}_r + \bar{\sigma}_\varphi = -EK_0(mr), \quad \bar{\tau}_{r\varphi} = 0; \ E = \frac{G(1+\mu)\alpha q}{\psi\lambda l}.$$

If we examine the instantaneous or continuous sources at the apex of the wedge, as follows from [6], the boundary conditions can be represented in the form of power series.

We can draw the conclusion that a number of problems relating to thermal elasticity for a wedge can be reduced to the basic problem of the theory of elasticity, if at the sides of the wedge we have distributed the surface forces that are symmetrical or antisymmetrical with respect to the axis of symmetry and if they are specified in the form of power series or in the form of certain generalizations. This is explained by the fact that the functions satisfying the equations of heat conduction are analytical with respect to the coordinates [5].

We solve the stated problem in general form.

Let surface forces specified in the form of converging series in the interval $(0, \infty)$ be distributed on the sides of the wedge:

$$(\sigma_\varphi + i\tau_{r\varphi})_{\varphi=\psi} = \sum_{n=0}^{\infty} r^n(\alpha_n \ln r + \beta_n), \tag{9}$$

$$(\sigma_\varphi + i\tau_{r\varphi})_{\varphi=-\psi} = \pm\sum_{n=0}^{\infty} r^n(\alpha_n \ln r + \beta_n). \tag{10}$$

In the following we will consider only the case of symmetrical loads, i.e., the plus sign is taken in (10). The solution for the antisymmetrical loads is achieved in analogous fashion.

We seek [3] the complex stresses in the form

$$\sigma_r + \sigma_\varphi = 2[\Phi(z) + \overline{\Phi(z)}], \tag{11}$$

$$\sigma_\varphi + i\tau_{r\varphi} = \Phi(z) + \overline{\Phi(z)} + e^{2i\varphi}[\bar{z}\Phi'(z) + \Psi(z)]. \tag{12}$$

We select the unknown analytical functions $\Phi(x)$ and $\Psi(z)$ in the form

$$\Phi(z) = \sum_{n=0}^{\infty} z^n (a_n \ln z + b_n), \quad \Psi(z) = \sum_{n=0}^{n} z^n (c_n \ln z + e_n). \tag{13}$$

Here we can assume that b_0 is a real number.

Considering the representations

$$\Phi(z) + \overline{\Phi(z)} = \sum_{n=0}^{\infty} \operatorname{Re}\{z^n (a_n \ln z + b_n)\}, \tag{14}$$

$$e^{2i\varphi}[\bar{z}\Phi'(z) + \Psi(z)] = \sum_{n=0}^{\infty} z^n \{(c_n \ln z + e_n) e^{2i\varphi} + a_n\} + \sum_{n=1}^{\infty} n z^n (a_n \ln z + b_n), \tag{15}$$

where $z = re^{i\varphi}$, $\ln z = \ln r + i\varphi$ (φ is the principal value of the argument) and the boundary conditions (9) and (10), respectively equating the coefficients for r^n and $r^n \ln r$, we obtain a system of equations for the unknowns a_n, b_n, c_n, and e_n. Following simple calculations, solution of this system yields

$$a_0 = \frac{\alpha_0}{2} - i\frac{\sin 2\psi \operatorname{Im} \beta_0}{\Delta_0}; \quad e_0 = \frac{2\psi i \operatorname{Im} \beta_0}{\Delta_0}; \quad b_0 = \frac{2\operatorname{Re} \beta_0 - \alpha_0}{4};$$

$$a_n = \sin(n+2)\psi \left[\frac{\operatorname{Re} \alpha_n}{\Delta_n^{(1)}} + i\frac{\operatorname{Im} \alpha_n}{\Delta_n^{(2)}}\right];$$

$$c_n \sin 2\psi = \bar{a}_n \sin 2n\psi - \alpha_n \sin n\psi, \quad c_0 = 0;$$

$$b_n = \frac{\operatorname{Re} A_n}{\Delta_n^{(1)}} + i\frac{\operatorname{Im} A_n}{\Delta_n^{(2)}};$$

$$A_n = \beta_n \sin(n+2)\psi + \alpha_n \psi \cos(n+2)\psi - 2\bar{a}_n \cos 2(n+1)\psi - a_n \sin 2\psi;$$

$$e_n \sin 2\psi = -\beta_n \sin n\psi + 2\bar{a}_n \psi \cos 2n\psi - \alpha_n \psi \cos 2\psi + \bar{b}_n \sin 2n\psi;$$

$$\Delta_n^{(1,2)} = (n+1)\sin 2\psi \pm \sin 2(n+1)\psi, \quad \Delta_0 = 2\psi \cos 2\psi - \sin 2\psi.$$

It follows from this system of equations that the number α_0 must be real. All of the coefficients under these assumptions are uniquely defined and calculated in sequence. Considering (14) and (15), separating the real and imaginary parts, the theoretical formulas for the stresses are written out in accordance with (11) and (12).

Let us consider the particularly important special case in which the load is specified at the side in the form of power series

$$\sigma_\varphi + i\tau_{r\varphi} = \sum_{n=0}^{\infty} \beta_n r^n \quad \text{when } \varphi = \pm\psi.$$

To find a solution for this problem, it must be assumed in the previous formulas that $\alpha_n = a_n = c_n = 0$, $a_0 \neq 0$. We find

$$b_n = \sin(n+2)\psi \left[\frac{\operatorname{Re} \beta_n}{\Delta_n^{(1)}} + i\frac{\operatorname{Im} \beta_n}{\Delta_n^{(2)}}\right], \quad a_0 = -i\frac{\sin 2\psi \operatorname{Im} \beta_0}{\Delta_0},$$

$$e_n \sin 2\psi = -\beta_n \sin n\psi + \bar{b}_n \sin 2n\psi, \quad e_0 = i\frac{2\psi \operatorname{Im} \beta_0}{\Delta_0}.$$

This case is of independent interest in the theory of elasticity, since the representation of the external loads in the form of power series is not burdensome in actual practice. This follows from the well-established fact that any smooth function can always be approximated by polynomials with whatever degree of accuracy is required.

In conclusion, we note that the uniform convergence of the series in the boundary conditions ensures uniform convergence of the series defining the complex functions and the stress components.

The case of the half-plane $\psi = \Pi/2$ must either be examined separately or the solution must be found from the general passage to the limit.

NOTATION

t	is the time;
μ, a, λ, h, α, G	are the Poisson coefficient, the coefficient of thermal diffusivity, the coefficient of thermal conductivity, the coefficient of linear expansion, the heat-transfer coefficient, and the shear modulus;
l	is the wedge thickness;
(r, φ), σ_{φ}, σ_r, $\tau_{r\varphi}$	are the coordinates of the points and of the stresses in a polar coordinate system, and the polar axis coincides with the wedge's axis of symmetry;
$K_0(r)$, $K_1(r)$	are cylindrical functions of the imaginary argument, of zeroth and first order;
i	is imaginary one;
$\mathrm{Re}\, z$, $\mathrm{Im}\, z$	are the real and imaginary parts of the complex number;

$\Delta = \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + (1/r^2)(\partial^2/\partial\varphi^2)$;

$m^2 = 2h/l\lambda$.

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